

The ratio of domination and independent domination numbers on trees

Shaohui Wang^{1,2*}, Bing Wei¹

1. *Department of Mathematics, The University of Mississippi,
University, MS 38677, USA*

2. *Department of Mathematics and Computer Science, Adelphi University,
Garden City, NY 11530, USA*

Abstract

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Let $\gamma(G)$ and $i(G)$ be the domination number and the independent domination number of G , respectively. In 1977, Hedetniemi and Mitchell began with the comparison of $i(G)$ and $\gamma(G)$ and recently Rad and Volkmann posted a conjecture that $i(G)/\gamma(G) \leq \Delta(G)/2$, where $\Delta(G)$ is the maximum degree of G . In this work, we prove the conjecture for trees and provide the graph achieved the sharp bound.

Keywords: Extremal graphs; Domination number; Independent domination number; Comparison.

1 Introduction

Throughout this paper $G = (V, E)$ is a simple undirected graph with vertex set $V(G)$ and edge set $E(G)$. For $v \in V(G)$, $N_G(v) = \{w \in V(G) : vw \in E(G)\}$ is the open neighborhood of v and $N_G[v] = N_G(v) \cup \{v\}$ is the closed neighborhood of v in G . If $N_G(v) = \emptyset$, v is called an isolated vertex. For $S \subseteq V(G)$, $N_G(S)$ is the open neighborhood of S , $N_G[S] = N_G(S) \cup S$ is the closed neighborhood of S and $G - S$ is a subgraph induced by $V(G) - S$. A graph F is a forest if it has no cycles. Specially, F is a tree if it contains only one component. A double star is a tree with exactly two vertices of degree greater than 1. In particular, if the two vertices have same degree,

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*Corresponding authors: S. Wang (e-mail: shaohuiwang@yahoo.com), B. Wei (e-mail: bwei@olemiss.edu).

then it is called a balanced double star. The line graph $L(G)$ of a connected graph is a graph such that each vertex of $L(G)$ represents an edge of G and two vertices of $L(G)$ are adjacent if and only if their corresponding edges share a common endpoint in G .

It is known that a vertex set $D \subset V(G)$ is a dominating set if every vertex of $V(G) - D$ is adjacent to some vertices of D . The minimum cardinality of a dominating set is called the domination number, denoted by $\gamma(G)$. Similarly, a vertex set $I \subset V(G)$ is an independent dominating set if I is both an independent set and a dominating set in G , where an independent set is a set of vertices in a graph such that no two of which are adjacent. The minimum cardinality of an independent dominating set is called the independent domination number, denoted by $i(G)$. Currently, lots of work relating domination number and independent domination number have been studied, referred to surveys [3, 5].

In 1977, S. Hedetniemi and S. Mitchell [6] showed that for any tree T , $\frac{i(L(T))}{\gamma(L(T))} = 1$, where $L(T)$ is the line graph of T . Because any line graph is a $K_{1,3}$ -free graph, R. B. Allan and R. Laskar [1] extended the previous result in 1978 and obtained that if a graph does not have an induced subgraph isomorphic to $K_{1,3}$, then $i(G)/\gamma(G) = 1$. Recently, Goddard et al.[4] considered the ratio $i(G)/\gamma(G)$ for regular graphs and proved that $i(G)/\gamma(G) \leq 3/2$ for cubic graphs. In 2013, Southey and Henning [8] improved the previous result to $i(G)/\gamma(G) \leq 4/3$ for connected cubic graphs except for $K_{3,3}$. During the same year, Rad and Volkmann [7] got an upper bound of $i(G)/\gamma(G)$ for a graph G and proposed the conjecture.

Theorem 1 (Rad and Volkmann [7]) *Let G be a graph, then*

$$\frac{i(G)}{\gamma(G)} \leq \begin{cases} \frac{\Delta(G)}{2}, & \text{if } 3 \leq \Delta(G) \leq 5, \\ \Delta(G) - 3 + \frac{2}{\Delta(G)-1}, & \text{if } \Delta(G) \geq 6. \end{cases}$$

Conjecture 2 (Rad and Volkmann [7]) *Let G be a graph with $\Delta(G) \geq 3$, then $i(G)/\gamma(G) \leq \Delta(G)/2$.*

In 2014, Furuta et al.[2] showed that $i(G)/\gamma(G) \leq \Delta(G) - 2\sqrt{\Delta(G)} + 2$ for a graph G and gave the graph achieved the new bound. However, when $\Delta(G)$ is big enough, then $\Delta(G) - 2\sqrt{\Delta(G)} + 2 > \Delta(G)/2$. Now there is a natural question that

Q: Is there other class of graphs, which has an affirmative answer for Conjecture 2?

Motivated by Conjecture 2 and the above question, we prove that Conjecture 2 is true for the tree and provide the graph G , which attains the sharp bound $\Delta(G)/2$.

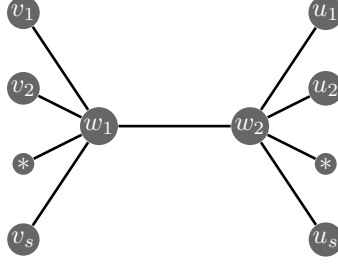


Figure 1: A balanced double star

Theorem 3 *Let G be a forest, then*

$$\frac{i(G)}{\gamma(G)} \leq \begin{cases} 1, & \text{if } \Delta(G) \leq 2, \\ \frac{\Delta(G)}{2}, & \text{if } \Delta(G) \geq 3, \end{cases}$$

and the equalities hold if either $\Delta(G) \leq 2$ or each component of G is a balanced double star (see figure 1).

As an immediate consequence of Theorem 3, we obtain that

Theorem 4 *Let G be a tree, then*

$$\frac{i(G)}{\gamma(G)} \leq \begin{cases} 1, & \text{if } \Delta(G) \leq 2, \\ \frac{\Delta(G)}{2}, & \text{if } \Delta(G) \geq 3, \end{cases}$$

and the equalities hold if either $\Delta(G) \leq 2$ or G is a balanced double star (see figure 1).

2 Proof of Theorem 3

In this section, we will prove Theorem 3 and start with an interesting lemma.

Lemma 1 *Let r_1, r_2, r_3, r_4, t be positive numbers with $\frac{r_1}{r_2} \leq t$ and $\frac{r_3}{r_4} \leq t$. Then $\frac{r_1+r_3}{r_2+r_4} \leq t$.*

Since $r_1 \leq r_2 t, r_3 \leq r_4 t$, we replace r_1, r_3 and obtain that Lemma 1 is true. Next we will give the main proof of this note.

Proof of Theorem 3. For $\Delta(G) \leq 1$, G contains only isolated vertices or edges and $i(G) = \gamma(G)$, that is, $i(G)/\gamma(G) = 1$. Next, we will consider the case of $\Delta(G) \geq 2$ and begin with the case that the forest G contains only one component, that is, G is a tree.

Let D be a minimum dominating set of G . Then $G[D]$ is also a forest. We build $\{G_i\}, \{x_i\}$ with $i \geq 1$ as follows: Let $G_1 = G[D]$ and $x_1 \in V(G_1)$

with $d_{G_1}(x_1) = 0$ or 1 ; For $i \geq 2$, if $V(G_i - N_{G_i}[x_i]) = \phi$, then stop and set $i = k$. Otherwise, let $G_i = G_{i-1} - N_{G_{i-1}}[x_{i-1}]$ and $x_i \in V(G_i)$ with $d_{G_i}(x_i) = 0$ or 1 .

Set $X = \{x_1, x_2, \dots, x_k\}$. Then X is an independent dominating set of $G[D]$ and $\{N_{G_i}[x_i], 1 \leq i \leq k\}$ is a partition of D , that is, $\sum_{1 \leq i \leq k} (d_{G_i}(x_i) + 1) = |D| = \gamma(G)$. Choose $I \subset V(G) - D$ such that $X \cup I$ is an independent dominating set of G , that is, $i(G) \leq |X| + |I| = k + |I|$. Since D is a dominating set of G , then $I = \cup_{v \in D-X} (N_G(v) \cap I) = \cup_{1 \leq i \leq k} (\cup_{v \in N_{G_i}(x_i)} (N_G(v) \cap I))$. By the choice of x_i , for $1 \leq i \leq k$ and $v \in N_{G_i}(x_i)$, we have $d_{G_i}(x_i) \leq d_{G_i}(v)$. Thus, $|N_G(v) \cap I| \leq d_G(v) - d_{G_i}(v) \leq \Delta(G) - d_{G_i}(x_i)$ and

$$\begin{aligned}
|I| &\leq \sum_{1 \leq i \leq k} \left(\sum_{v \in N_{G_i}(x_i)} |N_G(v) \cap I| \right) \\
&\leq \sum_{1 \leq i \leq k} \left(\sum_{v \in N_{G_i}(x_i)} (\Delta(G) - d_{G_i}(x_i)) \right) \\
&= \sum_{1 \leq i \leq k} \left(\sum_{v \in N_{G_i}(x_i)} \Delta(G) \right) - \sum_{1 \leq i \leq k} \left(\sum_{v \in N_{G_i}(x_i)} d_{G_i}(x_i) \right) \\
&= (|D| - k)\Delta(G) - \sum_{1 \leq i \leq k} d_{G_i}(x_i)^2. \tag{1}
\end{aligned}$$

By (1) and $|D| = \gamma(G)$, we can obtain that

$$\begin{aligned}
i(G) &\leq k + |I| \\
&\leq k + (|D| - k)\Delta(G) - \sum_{1 \leq i \leq k} d_{G_i}(x_i)^2 \\
&= \Delta(G)\gamma(G) - \sum_{1 \leq i \leq k} (\Delta(G) - 1 + d_{G_i}(x_i)^2),
\end{aligned}$$

that is,

$$\frac{i(G)}{\gamma(G)} \leq \Delta(G) - \frac{\sum_{1 \leq i \leq k} (\Delta(G) - 1 + d_{G_i}(x_i)^2)}{\gamma(G)}.$$

Now, it suffices to show that $-\frac{\sum_{1 \leq i \leq k} (\Delta(G) - 1 + d_{G_i}(x_i)^2)}{\gamma(G)} \leq -\frac{\Delta(G)}{2}$, that is,

$$\begin{aligned}
\sum_{1 \leq i \leq k} (\Delta(G) - 1 + d_{G_i}(x_i)^2) &\geq \frac{1}{2}\Delta(G)\gamma(G) \\
&= \frac{1}{2}\Delta(G) \left(\sum_{1 \leq i \leq k} (d_{G_i}(x_i) + 1) \right) \tag{2}
\end{aligned}$$

By the construction of G_i, x_i , $d_{G_i}(x_i) = d_{G_i}(x_i)^2 = 0$ or 1 . Thus, (2) is the same as (3) below.

$$\begin{aligned}
& \Leftrightarrow k\Delta(G) - k + \sum_{1 \leq i \leq k} d_{G_i}(x_i) - \frac{1}{2}\Delta(G)\left(\sum_{1 \leq i \leq k} d_{G_i}(x_i)\right) \\
& \quad - \frac{1}{2}\Delta(G)k \geq 0 \\
& \Leftrightarrow \left(1 - \frac{1}{2}\Delta(G)\right)\left(\sum_{1 \leq i \leq k} d_{G_i}(x_i) - k\right) \geq 0 \tag{3}
\end{aligned}$$

Furthermore, $d_{G_i}(x_i) = 0$ or 1 yields that $(\sum_{1 \leq i \leq k} d_{G_i}(x_i)) - k \leq 0$. Since $\Delta(G) \geq 2$, then $1 - \frac{1}{2}\Delta(G) \leq 0$. Thus, (3) is true, that is, Theorem 3 is true for the tree.

Next we will consider the case that G has more than one component. In this case, each component of G is either an isolated vertex or a tree, say G_1, G_2, \dots, G_s with an integer $s \geq 2$. For $1 \leq j \leq s$, if G_j is an isolated vertex, then $i(G_j)/\gamma(G_j) = 1/1 \leq \Delta(G)/2$; If G_j is a tree, by the above proof, $i(G_j)/\gamma(G_j) \leq \Delta(G)/2$. Finally, using Lemma 1, $i(G)/\gamma(G) \leq \Delta(G)/2$ holds for the forest. Furthermore, if $\Delta(G) \leq 2$, all forests achieve the bound; if $\Delta(G) \geq 3$, the union of balanced double stars attain the bound. Thus, Theorem 3 is true.

□

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